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A Localization Principle for Multiplicative Perturbations

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A certain resolvent estimate implies point spectrum for discrete linear wave equations with random local propagation speed. The significance of point spectrum in this context is that scattering within a random medium produces only localized standing waves. The implication is derived from a general result about multiplicative perturbations of a self-adjoint operator. The estimate implies point spectrum for almost every value of the perturbation parameter. © 1986 Academic Press, Inc.

1. Introduction

Wave propagation within a random medium is extraordinarily complicated. The wave is scattered and the scattered waves are themselves rescattered, and so on. It would seem that one should expect a diffusive behaviour. However in some circumstances there is a particularly dramatic effect: localization. This is when the scattered waves conspire to produce only standing waves, with no propagation throughout the medium.

In 1958 Anderson [1] predicted localization in random media with a large amount of disorder. Mott and Twose [12] observed in 1961 that localization is typical behavior in one dimensional random media, for every nonzero disorder. Goldsheid, Molchanov, and Pastur [9] proved in 1977 that the localization effect does indeed occur in one dimensional random media.

Progress in higher dimensions has been slower. The key contribution was the 1983 paper by Fröhlich and Spencer [7]. They derived estimates on the resolvent for a discrete random Schrödinger operator with large disorder, in any number of dimensions. The derivation of localization from these estimates is discussed in a number of recent papers, by Fröhlich, Martinelli, Scoppola, and Spencer [6], by Goldscheid [8], by Delyon, Lévy, and Souillard [5], and by Simon and Wolff [14]. The situation is reviewed in lecture notes by Carmona [3].

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The paper by Simon and Wolff contains a particularly elegant presentation, based on work of Aronszajn [2] and an idea of Kotani [10]. Their technique is to prove a result on rank one additive perturbations of a self-adjoint operator. The result is that a certain estimate on the resolvent implies point spectrum for almost every value of the perturbation parameter. This is what is needed for the application to random discrete Schrödinger operators. (Thomas and Wayne [16] have recently done related work on the general problem of additive perturbations of self-adjoint operators.) One advantage of this abstract approch is that the estimate on the resolvent need only be obtained at fixed frequency; no uniformity is required. This allows the application of the earlier fixed frequency estimates of Fröhlich and Spencer [7] instead of the uniform estimates of Fröhlich *et al.* [6].

The present paper applies the same strategy to rank one multiplicative perturbations. Section 2 describes the abstract setting. The resolvents have a more complicated dependence on the frequency and coupling constant parameters than in the case of additive perturbations. This results in a number of differences in the analysis; for instance the formula for the point masses is different. The perturbation result is presented in Section 3. This result depends on a new method of local averaging over the perturbation parameter. There is an application to a random discrete wave equation in the final Section 4. The equation is

$$m^2 \frac{\partial^2 w}{\partial t^2} = \Delta w, \tag{1}$$

where m is a random function of the space variable. The substitution $u = mw$ transforms this to self-adjoint form.

2. Multiplicative Perturbations

The most important function in the spectral theory of self-adjoint operators is the approximate delta function defined by

$$\delta_\varepsilon(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2} = \frac{1}{\pi} \Im \frac{1}{x - i\varepsilon} \tag{2}$$

for $\varepsilon > 0$. If μ is a finite positive measure defined on Borel subsets of the real line, then the functions

$$\int_{-\infty}^{\infty} \delta_\varepsilon(y - x)\, \mu(dy) = \frac{1}{\pi} \Im \int_{-\infty}^{\infty} \frac{1}{y - x - i\varepsilon}\, \mu(dy) \tag{3}$$

are densities of positive measures that converge weakly to μ as $\varepsilon \downarrow 0$. The question of pointwise convergence is more delicate. However, it is known [13] that these functions converge Lebesgue almost everywhere to a function $D\mu$ that is the density of the absolutely continuous part of the measure μ. Furthermore, the singular part of the measure μ is concentrated on the set where $D\mu = \infty$.

In the following another function will play an important role. This is

$$\Gamma(x) = \lim_{\varepsilon \downarrow 0} \frac{\pi}{\varepsilon} \int_{-\infty}^{\infty} \delta_\varepsilon(y-x)\, \mu(dy) = \int_{-\infty}^{\infty} \frac{1}{(y-x)^2}\, \mu(dy). \tag{4}$$

If $\Gamma(x) < \infty$, then $D\mu(x) = 0$. Furthermore it follows from the dominated convergence theorem that for each x for which $\Gamma(x) < \infty$

$$\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{1}{y-x-i\varepsilon}\, \mu(dy) = \int_{-\infty}^{\infty} \frac{1}{y-x}\, \mu(dy), \tag{5}$$

which is real.

As a consequence of these observations, if $\Gamma(x) < \infty$ for almost every x in some interval, then the limiting measure μ has no absolutely continuous part in this interval. It is possible, however, for it to have a singular part.

Another function of interest is the measure of $\mu(\{x\})$ of individual points. This is the other extreme; it is given by

$$\mu(\{x\}) = \lim_{\varepsilon \downarrow 0} \pi\varepsilon \int_{-\infty}^{\infty} \delta_\varepsilon(y-x)\, \mu(dy). \tag{6}$$

Clearly $\mu(\{x\}) > 0$ implies that $D\mu(x) = \infty$.

The relation of this to the spectral theory of self-adjoint operators is through the spectral theorem. Let H be a self-adjoint operator acting in a Hilbert space $\mathscr{H}$. The spectral theorem says that the spectral projections $1_S(H)$ are defined for each Borel subset S of the reals and project onto the subspace where $H \in S$. If ϕ is a vector in $\mathscr{H}$, then the measure

$$\mu(S) = \langle \phi, 1_S(H)\phi \rangle \tag{7}$$

is a finite positive measure. The approximating densities in this case are

$$\langle \phi, \delta_\varepsilon(H-x)\phi \rangle = \frac{1}{\pi} \Im \langle \phi, (H-z)^{-1}\phi \rangle, \tag{8}$$

where $z = x + i\varepsilon$.

In this context the function Γ is

$$\Gamma(x) = \langle \phi, (H-x)^{-2}\phi \rangle. \tag{9}$$

Thus if $\Gamma(x) < \infty$ and $z = x + i\varepsilon$, then

$$\lim_{\varepsilon \downarrow 0} \langle \phi, (H-z)^{-1}\phi \rangle = \langle \phi, (H-x)^{-1}\phi \rangle, \tag{10}$$

which is real. The limit $D\mu(x)$ of the imaginary part vanishes at x.

If $\Gamma(x) < \infty$ for almost every x in some interval I, then the measure has no absolutely continuous part on this interval. However, it is possible that H has a countable dense set of eigenvalues in this interval, since the eigenvalues form a set of measure zero. The unit vector ϕ must be chosen so that its inner products with the corresponding eigenvectors approach zero sufficiently rapidly. It is also possible that H has singular continuous spectrum in this interval.

From now on we consider a fixed Hilbert space $\mathscr{H}$. The extra ingredients are a positive self-adjoint operator H_1 acting in $\mathscr{H}$ and a unit vector ϕ in the domain of H_1.

DEFINITION 1. Let P be the orthogonal projection onto the one-dimensional space spanned by the unit vector ϕ, so that

$$P\psi = \phi\langle \phi, \psi \rangle. \tag{11}$$

The *multiplicative perturbation operators* B_λ are defined for $\lambda > 0$ by

$$B_\lambda = 1 - P + \lambda P. \tag{12}$$

The *multiplicatively perturbed operators* H_λ are defined to be

$$H_\lambda = B_\lambda^{-1} H_1 B_\lambda^{-1}, \tag{13}$$

where again we require that $\lambda > 0$.

Note that B_λ is equal to λ in the direction of ϕ and is equal to 1 on the orthogonal complement. Thus it is easy to see that B_λ is invertible and that $B_\lambda^{-1} = B_{\lambda^{-1}}$.

The key to further analysis is the resolvent identity

$$\begin{aligned}(H_\lambda - z)^{-1} &= B_\lambda (H_1 - B_\lambda^2 z)^{-1} B_\lambda \\ &= B_\lambda (H_1 - z)^{-1} B_\lambda + B_\lambda (H_1 - z)^{-1} (B_\lambda^2 - 1)\, z (H_1 - B_\lambda^2 z)^{-1} B_\lambda.\end{aligned} \tag{14}$$

This gives almost immediately the crucial *perturbation identity*:

$$\langle \phi, (H_\lambda - z)^{-1}\phi \rangle = \frac{\lambda^2 \langle \phi, (H_1 - z)^{-1}\phi \rangle}{1 - (\lambda^2 - 1)\, z \langle \phi, (H_1 - z)^{-1}\phi \rangle}. \tag{15}$$

In other words, the diagonal matrix element $w = \langle \phi, (H_1 - z)^{-1}\phi \rangle$ is sent into the diagonal matrix element $w_\lambda = \langle \phi, (H_\lambda - z)^{-1}\phi \rangle$ by

$$w_\lambda = \frac{\lambda^2}{(1/w) - (\lambda^2 - 1)z}. \tag{16}$$

3. Perturbed Spectra

The following two perturbation theorems are the main abstract results. The first says that a certain estimate on the unperturbed operator implies that the spectral measure of the perturbed operator typically has no absolutely continuous part. The second says that the same estimate implies that it also typically has no singular continuous part. The only remaining possibility is point masses. These results are valid for typical values of the perturbation parameter. There may of course be exceptional values as well. The notation is as in the previous section.

An open interval of the positive real axis that is bounded away from zero and from infinity will be called a *positive interval*. The hypothesis in both theorems is that $\Gamma_1(x) = \langle \phi, (H_1 - x)^{-2}\phi \rangle < \infty$ for Lebesgue almost every x in a positive interval I. One can hope to prove such an estimate by approximating H_1 by an operator known to have point spectrum and estimating the influence of the eigenvalues near x in I [7].

Theorem 1. *Let H_λ, $\lambda > 0$, be multiplicatively perturbed operators. Let I be a positive interval. Assume that*

$$\Gamma_1(x) = \langle \phi, (H_1 - x)^{-2}\phi \rangle < \infty \tag{17}$$

for Lebesgue almost every x in I. Then for Lebesgue almost every $\lambda > 0$, the measure μ_λ given by $\mu_\lambda(S) = \langle \phi, 1_S(H_\lambda)\phi \rangle$ has no absolutely continuous part in I.

Proof. We work in the strip consisting of all x in the interval I and all $\lambda > 0$. Let

$$\begin{aligned} G &= \{x, \lambda \mid \Gamma_1(x) = \infty\}, \\ Q &= \{x, \lambda \mid x \notin G, (\lambda^2 - 1)\, x \langle \phi, (H_1 - x)^{-1}\phi \rangle = 1\}. \end{aligned} \tag{18}$$

Consider x, λ in $G^c \cap Q^c$. Then the boundary value of $\langle \phi, (H_1 - z)^{-1}\phi \rangle$ with $z = x + i\varepsilon$ as $\varepsilon \downarrow 0$ is real. In addition, it follows from the perturbation identity (15) that the boundary value of $\langle \phi, (H_\lambda - z)^{-1}\phi \rangle$ is also real and therefore

$$D\mu_\lambda(x) = \lim_{\varepsilon \downarrow 0} \langle \phi, \delta_\varepsilon(H_\lambda - x)\phi \rangle = 0. \tag{19}$$

Let

$$A = \{x, \lambda \mid D\mu_\lambda(x) \neq 0\}. \tag{20}$$

We have just seen that the point x, λ is in A^c. This proves that A is contained in $G \cup Q$.

The hypothesis of the theorem immediately implies that G has planar Lebesgue measure zero. On the other hand, for each x there are at most two λ for which the equation defining Q is satisfied. Therefore the planar Lebesgue measure of Q is also zero. This proves in particular that the planar Lebesgue measure of A is zero. ∎

THEOREM 2. *Let H_λ, $\lambda > 0$, be multiplicatively perturbed operators. Let I be a positive interval. Assume that*

$$\Gamma_1(x) = \langle \phi, (H_1 - x)^{-2}\phi\rangle < \infty \tag{21}$$

for Lebesgue almost every x in I. Then for Lebesgue almost every $\lambda > 0$, the measure μ_λ given by $\mu_\lambda(S) = \langle \phi, 1_S(H_\lambda)\phi\rangle$ has no singular continuous part in I.

Proof. Let

$$\begin{aligned} S_\lambda &= \{x \in I \mid D\mu_\lambda(x) = \infty\}, \\ N_\lambda &= \{x \in I \mid \mu_\lambda(\{x\}) = 0\}. \end{aligned} \tag{22}$$

The singular continuous part of the measure μ_λ is concentrated on $S_\lambda \cap N_\lambda$. Let

$$G = \{x \in I \mid \Gamma_1(x) = \infty\}. \tag{23}$$

Let x be in $S_\lambda \cap G^c$. Assume $\lambda \neq 1$. Then the denominator in the fundamental perturbation identity (15) vanishes. The formula for the point masses is

$$\mu_\lambda(\{x\}) = \langle \phi, 1_{\{x\}}(H_\lambda)\phi\rangle = \lim_{\varepsilon \downarrow 0} \langle \phi, -i\varepsilon(H_\lambda - x - i\varepsilon)^{-1}\phi\rangle. \tag{24}$$

The limit may be calculated by using the perturbation identity and L'Hospital's rule. At points where the denominator vanishes the limit is

$$\mu_\lambda(\{x\}) = \frac{\lambda^2}{(\langle \phi, (H_1 - x)^{-2}\phi\rangle / \langle \phi, (H_1 - x)^{-1}\phi\rangle^2) + (\lambda^2 - 1)}. \tag{25}$$

This is clearly between zero and 1. In fact, since

$$\Gamma_1(x) = \langle \phi, (H_1 - x)^{-2}\phi\rangle < \infty, \tag{26}$$

we may conclude that $\mu_\lambda(\{x\})>0$. The conclusion is that $x\in N_\lambda^c$. This proves that $S_\lambda \cap N_\lambda$ is contained in G.

The set G has Lebesgue measure zero and is independent of λ, but it does not follow from this alone that there is no singular continuous spectrum! The crucial step is a process of averaging over λ [14]. In our argument we take local averages over an interval (a, b). Let

$$\eta = \int_a^b \mu_\lambda \frac{d\lambda}{\lambda}. \tag{27}$$

Then the perturbation identity gives

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{y-z}\,\eta(dy) &= -\frac{1}{2z}\int_a^b \frac{2\lambda}{(\lambda^2-1)-1/(z\langle\phi,(H_1-z)^{-1}\phi\rangle)}\,d\lambda \\ &= \frac{1}{2z}\log\left(\lambda^2-1-\frac{1}{z\langle\phi,(H_1-z)^{-1}\phi\rangle}\right)\Bigg|_{\lambda=b}^{\lambda=a}.\end{aligned} \tag{28}$$

At this point it is useful to note certain properties of the integrand in the λ integral above. Observe that the product

$$z\langle\phi,(H_1-z)^{-1}\phi\rangle = \langle\phi, H_1(H_1-z)^{-1}\phi\rangle - 1.$$

Since H_1 is a positive operator and z is in the upper half plane, this product also lies in the upper half plane. This gives control of the singularities of the integrand as a function of λ. It follows easily that the imaginary part of the difference of the logarithms is between 0 and π.

We have shown that

$$\int_{-\infty}^{\infty} \frac{z}{y-z}\,\eta(dy) = g(z), \tag{29}$$

where the imaginary part of $g(z)$ is bounded for z in the upper half plane. This implies that

$$\int_{-\infty}^{\infty} \delta_\varepsilon(x-y)\; y\eta(dy) = \frac{1}{\pi}\,\Im g(x+i\varepsilon). \tag{30}$$

Let $\varepsilon \downarrow 0$. The function of x on the left defines a measure that converges weakly to the measure $x\eta$. Since the functions of x on the right are uniformly bounded, this limiting measure must be given by a bounded function.

Recall that the measure η was defined as an average of the measures μ_λ. We have just shown that the averaged measure is absolutely continuous with respect to Lebesgue measure on I. However, the singular continuous spectrum in I is supported on a set G of Lebesgue measure zero. Therefore the η measure of G is also zero. It follows that the μ_λ measure of G is also zero for almost every λ in the interval (a, b). This is enough to imply the absence of singular continuous spectrum for these λ. ∎

4. Random Discrete Wave Equations

Let H be a positive self-adjoint operator acting in the Hilbert space $\mathscr{H}$. The corresponding wave equation is

$$\frac{d^2u}{dt^2} + Hu = 0. \tag{31}$$

It is well known that the initial value problem for this equation is well posed in a certain auxiliary Hilbert space defined in terms of an energy norm.

In the kind of application we have in mind, the Hilbert space is $\mathscr{H} = l^2(\mathbf{Z}^\nu)$, the space of functions f defined on the ν dimensional integer lattice $\mathbf{Z}^\nu$ and satisfying

$$\sum_{r \in \mathbf{Z}^\nu} |f(r)|^2 < \infty. \tag{32}$$

The operator H is defined by

$$H = -M^{-1} \Delta M^{-1}, \tag{33}$$

where Δ is the finite difference Laplacian, and where M is multiplication by a positive function m on $\mathbf{Z}^\nu$ that is bounded away from zero and from infinity.

To make the operator random, we take the values $m(\mathbf{r})$ of the function m at the integer points $\mathbf{r}$ in $\mathbf{Z}^\nu$ to be independent and identically distributed random variables.

THEOREM 3. *Let $H = -M^{-1} \Delta M^{-1}$ be the random operator defining the random discrete wave equation. Assume that the common distribution of the random variables $m(\mathbf{r})$ is absolutely continuous with respect to Lebesgue measure. Let I be a positive interval. Let ϕ_0 be the function on $\mathbf{Z}^\nu$ that is 1 at the origin and zero elsewhere. Assume that with probability one*

$$\langle \phi_0, (H-x)^{-2} \phi_0 \rangle < \infty \tag{34}$$

for Lebesgue almost every x in I. Then with probability one H has only point spectrum in I.

Proof. Fix a realization of the random variables $m(\mathbf{r})$ and hence of the operator H. Let $H_1 = H$ and define H_λ as in the previous sections. Let $m(*)$ be another random variable with the common absolutely continuous distribution, independent of all the others. Make λ a random variable by defining it to be $\lambda = m(*)/m(\mathbf{0})$. Since the distribution of λ is continuous, it follows from Theorems 1 and (2) that with probability one the measure μ_λ associated with H_λ and ϕ_0 has only point masses.

In the original random problem H_λ with random λ is precisely the original random H, but with $m(\mathbf{0})$ replaced by $m(*)$. But this is an isomorphic random operator. Thus the random measure μ associated with H and ϕ_0 has only point masses.

Since the whole problem is translation invariant, the same is true when the unit vector is taken to be the function $\phi_\mathbf{r}$ defined to be one at $\mathbf{r}$ in $\mathbf{Z}^\nu$ and zero elsewhere. Such vectors form a basis for $\mathscr{H} = l^2(\mathbf{Z}^\nu)$, so we may conclude that the random operator H has only point spectrum. ∎

The interest of the above theorem is that the hypothesis (34) may be satisfied on an interval that is also an interval of spectrum of H. This means that on this interval of frequency the discrete wave equation in a random medium has only standing waves, in sharp contrast to the case of localized disturbances in a constant medium, where scattering behaviour is known to occur. There is a dense set of eigenvalues, and each eigenvalue is associated with an eigenfunction that is approximately localized in some bounded region of the space $\mathbf{Z}^\nu$. If x is fixed, then with probability one it is not an eigenvalue. It will be a limit point of eigenvalues, but they will correspond to eigenfunctions localized in increasingly remote regions of $\mathbf{Z}^\nu$.

The easy part is to identify the spectrum of H. Let a be in the essential range of the random variables $1/m(\mathbf{r})$. Then there will be arbitrarily large regions in the space $\mathbf{Z}^\nu$ on which all the $1/m(\mathbf{r})$ for $\mathbf{r}$ in the region are close to a. On such a region H resembles the operator $-a^2\Delta$, which has spectrum $[0, 4\nu a^2]$. Furthermore H will have approximate eigenfunctions localized in the interior of this region with eigenvalues close to any point in this interval. Thus the spectrum of H includes the interval $[0, 4\nu a^2]$. This sort of argument may be made into a proof [11]. The conclusion is that if the density of the $1/m(\mathbf{r})$ has even a small tail at high frequency, then there will be spectrum at high frequency.

The real question is how to obtain the estimate (34). The discrete random wave equation has been thoroughly investigated in the case $\nu = 1$ [4]. The estimate is satisfied at all nonzero frequencies. In fact, it is easy to obtain enough uniformity to give a direct proof of the existence of dense point spectrum.

The question is more subtle in dimensions $\nu \geqslant 2$. It is believed by physicists [15] that at least for $\nu > 2$ there is transition between continuous spectrum and point spectrum. For the random wave equation point spectrum is expected at high frequencies. It should be possible to prove this by imitating the argument of Fröhlich and Spencer [7].

The strategy is to pick a frequency x so high that the regions where $4v/m(\mathbf{r})^2$ ever exceeds x are sparse. If we wall off these regions with Dirichlet boundary conditions, then it is easy to estimate the resolvent in the remaining region. In fact, this remaining region is a "forbidden region" for wave propagation at frequency x.

The remaining part of the argument is to successively reintroduce bounded regions where the spectrum gets close to the frequency x. The resolvent is controlled in these regions by a lemma on the density of eigenvalues. In the regime of interest these regions of near resonance are very infrequent, and so their influence can be controlled.

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